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# A note on $c = 1$ Virasoro boundary states and asymmetric shift orbifolds

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Abstract: We comment on the conformal boundary states of the $c = 1$ free boson theory on a circle which do not preserve the U(1) symmetry. We construct these Virasoro boundary states at a generic radius by a simple asymmetric shift orbifold acting on the fundamental boundary states at the self-dual radius. We further calculate the boundary entropy and find that the Virasoro boundary states at irrational radius have infinite boundary entropy. The corresponding open string description of the asymmetric orbifold is given using the quotient algebra construction. Moreover, we find that the quotient algebra associated with a non-fundamental boundary state contains the noncommutative Weyl algebra.

Keywords: D-branes, Conformal and W Symmetry, Non-Commutative Geometry, Boundary Quantum Field Theory.





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## 1. Introduction

Recently, there has been renewed interest in the conformal boundary states of theories with central charge c=1 [1]–[4]. For the c=1 free boson theory taking values on a circle with arbitrary radius $R$, there are two well-known classes of boundary states corresponding to Dirichlet and Neumann boundary conditions. Considering only one spatial direction in the string theory framework, these conformal boundary conditions correspond to D0-branes and D1-branes, respectively. Each brane has a classical modulus - for the D0-brane, the value of its location, and for the D1-brane, the value of the Wilson line.

It has also been known that there exist additional conformal boundary states for $R = M$ or $1/M$, integer multiples (or one over integer multiples) of the self-dual radius, $R_{sd} = \sqrt{\alpha'} = 1$ [5]–[8]. These conformal boundary conditions can not be simply expressed using only the left and right U(1) currents. They correspond in string theory to the addition of a marginal tachyon potential on the boundary of the worldsheet action. The marginal tachyon potential breaks the extended U(1) current algebra but still preserves the Virasoro algebra. Recent works have now given explicit constructions of non-U(1) conformal boundary states for any rational [2] and irrational radius [2, 3].

Central to the construction of all non-U(1) boundary states is the presence of the non-U(1) discrete state primaries in the c=1 CFT (see [9] for details and references). The U(1) representation with the highest weight primary $e^{ipX(z)}$ is reducible under the Virasoro algebra when the conformal dimension $h = p^2/4 = j^2$ where $j \in \mathbb{Z}/2$ (and similarly for the anti-holomorphic sector). Thus, for integer values of $p$, there are U(1) descendants that are Virasoro primaries. Given that the left and right momenta at $R = 1$ have values $(p_L, p_R) = (n+m, n-m)$ for $m, n \in \mathbb{Z}$, all discrete primaries are found in the $R = 1$ self-dual theory and are organized into multiplets of the SU(2)×SU(2) enhanced symmetry. These discrete primaries and their descendants can then be grouped together to form Virasoro Ishibashi states. The non-U(1) boundary states are then a linear combination of Virasoro Ishibashi states with coefficients chosen to satisfy Cardy's condition [10]. We will call all $c = 1$ conformal boundary states that can only be constructed using Virasoro Ishibashi states Virasoro boundary states.

Due to the discreteness of the momenta $(p_L, p_R) = (n/R+mR, n/R-mR)$, theories at different radii have different sets of Virasoro Ishibashi states. However, it is important to emphasize that all possible Virasoro Ishibashi states are present in the $R = 1$ theory. This suggests that there may be a connection between the Virasoro boundary states constructed at $R \neq 1$ to those at $R = 1$. Below we will show that the connection at a generic radius $r$ is that of an asymmetric shift orbifold. With the shift acting differently on the left- and right-movers, the closed string theory at $R = 1$ can be orbifolded to the closed string theory at any other radius. Thus, all Virasoro boundary states at $R \neq 1$ can be obtained by the orbifold boundary state construction. Effectively, the orbifold action on the $R = 1$ boundary states projects out Virasoro Ishibashi states in the $R = 1$ theory that are not present in the $R = r$ theory.

In section 2, we introduce our notation by reviewing the construction of $R = 1$ Virasoro boundary states. We present in section 3 the asymmetric shift orbifold and the boundary state orbifold construction connecting Virasoro boundary states at different radii. We then proceed to analyze the orbifold boundary states for their algebraic and geometrical significance. In section 4, we describe the boundary state orbifold operation in the open string framework. Here, the analogous prescription is given by the quotient algebra construction. For the D1- and D0-brane boundary conditions at a generic radius, we note the presence of the noncommutative Weyl algebra in the quotient algebra. In section 5, we conclude with a discussion generalizing the orbifold construction of branes and propose a condition for establishing a connection between boundary field theories situated within different bulk theories.



## 2. $R = 1$ Virasoro boundary states

Before generating the Virasoro boundary states for $R \neq 1$, we introduce our notation by writing down the U(1) and Virasoro boundary states at $R = 1$ (see [8, 11] for some of the details). As mentioned above, the Virasoro primaries are organized into SU(2)×SU(2) multiplets with the SU(2) currents given by

$$J^x(z) = \cos 2X_L(z)\,, \qquad J^y(z) = \sin 2X_L(z)\,, \qquad J^z(z) = i\partial X_L(z)\,, \tag{2.1}$$

in the holomorphic sector (and similarly in the anti-holomorphic sector). For constructing boundary states, we are mainly interested in the spin zero primaries. These can be generated from the highest weight states $e^{i\,2j(X_L(z)+X_R(\bar z))}$ where $X(z,\bar z) \equiv X_L(z) + X_R(\bar z)$ and

$j = 0, 1/2, 1, \ldots$ Acting with lowering operators $J_0^-(z)$ and $\tilde{J}_0^-(\bar{z})$, the spin zero primaries are explicitly

$$\phi^j_{m,n}(z,\bar{z}) = \frac{1}{(2j)!}\sqrt{\frac{(j+m)!(j+n)!}{(j-m)!(j-n)!}}\left[\oint_{\mathcal{C}}\frac{dw}{2\pi i}\, e^{-i\,2X_L(w)}e^{i\frac{\pi}{2}\hat{p}_R}\right]^{j-m}\times$$
$$\times\left[-\oint_{\mathcal{C}}\frac{d\bar{w}}{2\pi i}\, e^{-i\,2X_R(\bar{w})}e^{-i\frac{\pi}{2}\hat{p}_L}\right]^{j-n} e^{i\,2j(X_L(z)+X_R(\bar{z}))}e^{i\frac{\pi}{2}j(\hat{p}_L-\hat{p}_R)}\,, \quad (2.2)$$

where we have included the appropriate normalization factor and also cocycle factors which have dependence on the left and right momentum operators, $\hat{p}_L = 2J_0^z$ and $\hat{p}_R = 2\tilde{J}_0^z$, respectively. The cocycle factor $c_k(\hat{p}) = e^{i\frac{\pi}{4}(k_R\hat{p}_L - k_L\hat{p}_R)}$ is attached to the right of each vertex operator with zero mode $e^{ik_LX_L(z)+ik_RX_R(\bar{z})}$. The operator $c_k(\hat{p})$ satisfies

$$c_k(\hat{p}+k')c_{k'}(\hat{p}) = (-1)^{\frac{1}{2}(k_Lk'_L - k_Rk'_R)}c_{k'}(\hat{p}+k)c_k(\hat{p}) = \epsilon(k,k')c_{k+k'}(\hat{p})\,, \quad (2.3)$$

with the two-cocycle $\epsilon(k,k') = e^{i\frac{\pi}{4}(k_Rk'_L - k_Lk'_R)}$. The cocycles are needed to insure that the vertex operators commute amongst themselves [12]. We point out that the expression for the cocycle factor is not unique. Our choice of $c_k(\hat{p})$ preserves the commutation relations of the SU(2) × SU(2) current algebra.[1] Besides the cocycle, the primaries $\phi^j_{m,n}(z,\bar{z})$ in (2.2) have $h = \tilde{h} = j^2$ and are polynomials in derivatives of $X$ multiplied by the zero mode, $e^{i\,2mX_L + i\,2nX_R}$. We will label the Virasoro Ishibashi states generated by the primaries $\phi^j_{m,n}(z,\bar{z})$ by $|j;m,n\rangle\rangle$.

The Dirichlet D0-brane boundary state is typically written as a sum over U(1) Ishibashi states with $p_L = p_R$. In particular, a D0-brane located at $x_0 = 0$ for $R = 1$ can be expressed as

$$|D0\rangle\rangle_{x_0=0} = \frac{1}{2^{\frac{1}{4}}}\sum_{n=-\infty}^{\infty}\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\alpha_{-m}\tilde{\alpha}_{-m}\right)|p_L = n, p_R = n\rangle$$
$$= \frac{1}{2^{\frac{1}{4}}}\sum_{j,m}(-1)^{j-m}|j;m,m\rangle\rangle\,, \quad (2.4)$$

where in the second line we have written the boundary state as a sum over Virasoro Ishibashi states.[2,3] Taking the string worldsheet to be the upper half-plane, the D0-brane at $x_0 = 0$ implies the following boundary conditions for the SU(2) currents at the boundary:

$$J^x(z) = \tilde{J}^x(\bar{z})|_{z=\bar{z}}\,, \qquad J^y(z) = -\tilde{J}^y(\bar{z})|_{z=\bar{z}}\,, \qquad J^z(z) = -\tilde{J}^z(\bar{z})|_{z=\bar{z}}\,. \quad (2.5)$$

These conditions arise from the general Dirichlet boundary condition $X_L = -X_R + x_0$ at $z = \bar{z}$. As operators acting on $|D0\rangle\rangle_{x_0=0}$, the boundary conditions (2.5) can be expressed

[1] For additional details on two-cocycles, see [13, 14, 15].

[2] Note that the presence of cocycle changes the definition of $\phi^j_{m,n}(z,\bar{z})$ up to a phase. This becomes important in matching the U(1) and Virasoro primaries in (2.4) and checking the D0-brane boundary conditions.

[3] Some authors, as for example in [6], normalize $|j;m,n\rangle\rangle$ so that the factor $(-1)^{j-m}$ is absent in the second line of (2.4). Our definition of $|j;m,n\rangle\rangle$ allows us to use standard formulas for operators in SU(2) representations.

for all integer $n$ as

$$\left(e^{iJ_0^x\pi}J_n^a e^{-iJ_0^x\pi}+\tilde{J}_{-n}^a\right)\ |D0\rangle\rangle_{x_0=0}=0\,. \tag{2.6}$$

All fundamental boundary states at the self-dual radius, including $|D0\rangle\rangle_{x_0=0}$ , contain three truly marginal boundary fields. These fields can be used to deform the theory and map out the space of boundary conditions at $R=1$. Under the deformation,

$$|D0\rangle\rangle_{x_0=0}\to|g\rangle\rangle=e^{i\vec{J}_0\cdot\hat{n}\,\phi}\ |D0\rangle\rangle_{x_0=0}=\frac{1}{2^{\frac{1}{4}}}\sum_{j,m,n}(-1)^{j-n}\,D^j_{m,n}(g)|j;m,n\rangle\rangle\,. \tag{2.7}$$

The moduli space of fundamental boundary states is labelled by $g\in\mathrm{SU}(2)$ and $D^j_{m,n}(g)$ in (2.7) is the matrix element of $g$ in the $j^{\text{th}}$ representation.[4] We will conveniently parameterize $g$ by

$$g(\lambda,\phi_1,\phi_2)=\begin{pmatrix}\cos\lambda\ e^{i\phi_1} & -i\sin\lambda\ e^{-i\phi_2}\\ -i\sin\lambda\ e^{i\phi_2} & \cos\lambda\ e^{-i\phi_1}\end{pmatrix}, \tag{2.8}$$

where $\phi_1$ and $\phi_2$ are periodic in $2\pi$ and $0\le\lambda\le\pi/2$.

The fundamental boundary state $|g\rangle\rangle$ satisfies

$$\mathrm{Ad}(g\cdot i)J_n^a+\tilde{J}_{-n}^a\ |g\rangle\rangle=0\qquad\text{with}\qquad i=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \tag{2.9}$$

By working out the explicit boundary conditions as in (2.5) for $|D0\rangle\rangle_{x_0=0}$ , one finds that at $\lambda=0$, the boundary state satisfies the Dirichlet condition, $J^z(z)=-\tilde{J}^z(\bar{z})|_{z=\bar{z}}$, with the D0-brane situated at $x_0=R\phi_1=\phi_1$. And at $\lambda=\pi/2$, the boundary state satisfies the Neumann condition, $J^z(z)=\tilde{J}^z(\bar{z})|_{z=\bar{z}}$, with the value of the constant gauge potential on the D1-brane (in units of length) given by $\tilde{x}_0=\frac{\alpha'}{R}\phi_2=\phi_2$. Here, the Neumann boundary condition with a constant gauge potential corresponds to the worldsheet boundary condition $X_L=X_R+\tilde{x}_0$ at $z=\bar{z}$. In short, we have

$$|D0\rangle\rangle_{x_0}=|g(\lambda=0,\phi_1=x_0)\rangle\rangle\quad\text{ and }\quad|D1\rangle\rangle_{\tilde{x}_0}=\left|g\left(\lambda=\frac{\pi}{2},\phi_2=\tilde{x}_0\right)\right\rangle\!\!\Big\rangle, \tag{2.10}$$

where we have noted the independence of $g$ on the parameters $\phi_2$ and $\phi_1$ when $\lambda=0$ and $\lambda=\pi/2$, respectively. The parameter $\lambda$ is physically the renormalized value of the strength of the marginal tachyon potential [6]. And finally, the Virasoro boundary states at $R=1$ are the boundary states $|g(\lambda\neq0\,\text{or}\,\pi/2,\phi_1,\phi_2)\rangle\rangle$.

## 3. Boundary state orbifold from $R=1$ to $R=r$

Gaberdiel and Recknagel in [2] constructed the Virasoro boundary states at $R=M/N$ starting with the Virasoro boundary states at $R=1$ and then projecting out those Virasoro Ishibashi states not present at the rational radius. We will motivate their projection physically as arising from the construction of boundary states in an orbifold theory.

[4] In [11], it was found that the moduli space at $R=1$ can be at most $\mathrm{SL}(2,\mathbb{C})$. We will not consider these possible additional fundamental boundary states here.

### 3.1 Closed string orbifold to $R = r$

Before proceeding to orbifolding the boundary states, we first identify the closed string orbifold connecting the theory at $R = 1$ to that at a generic radius $R = r$. The closed string theory at any rational radius can be obtained from that at $R = 1$ by a combination of two symmetric $\mathbb{Z}_N$ shift orbifolds and a T-duality. Explicitly, starting at $R = 1$, we apply a $\mathbb{Z}_M$ shift orbifold, $X(z,\bar{z}) \equiv X_L(z) + X_R(\bar{z}) \to X + 2\pi/M$ to reduce the radius to $R = 1/M$. After a T-duality, we orbifold again by $\mathbb{Z}_N$ to obtain the closed string theory at radius $R = M/N$. However, we would like to consider the procedure as a single orbifold. The only obstacle is the presence of a T-duality and it can be overcome by considering orbifolding the dual field $\tilde{X}(z,\bar{z}) \equiv X_L(z) - X_R(\bar{z})$. Take for example the "orbifold" to the $R = 2$ theory. The desired orbifold must project out half of the winding states, those with vertex operator of the form $e^{i(2m+1)(X_L - X_R)}$, and generate in the twisted sectors new momentum modes, $(p_L, p_R) = (n+1/2, n+1/2)$. By recalling the mode expansion at radius $R$ for $\tilde{X}(\tau,\sigma) = X_L(z) - X_R(\bar{z}) = \frac{x_L}{2} - \frac{x_R}{2} + \frac{n}{R}\sigma +$ *oscillator modes*, it becomes evident that the orbifold is constructed by a shift of the dual coordinate $\tilde{X} \to \tilde{X} + 2\pi/r$ for $r = 2$. In particular, the orbifold can be written explicitly as $X_L \to X_L + \pi/2$ and $X_R \to X_R - \pi/2$; thus, it is a simple example of an asymmetric orbifold [16].

Now generalizing the orbifold to arbitrary rational radius $r$, we find that the required asymmetric orbifold group action $\Gamma$ is generated by the following two elements: $X \to X + 2\pi r$ and $\tilde{X} \to \tilde{X} + 2\pi/r$. Physically, the generators reset the periodicities of the coordinate and the dual coordinate to that required at the $R = r$ theory. (Recall that at $R = 1$, both $X$ and $\tilde{X}$ have periodicities of $2\pi$.) We will label a group element $h \in \Gamma$ by two indices $(m', n')$ with action on the fields given by

$$h_{m',n'}: \quad X \to X + 2\pi r m' , \qquad \tilde{X} \to \tilde{X} + 2\pi n'/r . \tag{3.1}$$

The identity element is $h_{0,0}$ and the order of the group action is given by $|\Gamma| \equiv (m'_{\max} + 1)$ $(n'_{\max} + 1)$.

The orbifold group action $\Gamma = \mathbb{Z}_{(m'_{\max}+1)} \times \mathbb{Z}_{(n'_{\max}+1)}$ with elements given in (3.1) can in fact be applied to any irrational radius $r$. The only difference is that $|\Gamma|$ is now infinite with both $m'_{\max}$ and $n'_{\max}$ taken to infinity. The orbifold theory is the closed string theory at the irrational radius $r$. Here, only states with zero left and right momenta in the $R = 1$ theory are invariant under the orbifold projection. The momentum mode $(p_L, p_R) = (n'/r + m'r, n'/r - m'r)$ and its conformal descendants are generated by the twisted sector associated with $h_{m',n'}$. Thus, setting $m'_{\max} = n'_{\max} = \infty$, the orbifold partition function will contain all the left and right momentum combinations of the $R = r$ theory. All together, we have obtained an asymmetric shift orbifold that allows us to orbifold the $R = 1$ closed string theory to that at any arbitrary radius. In the following, for ease of description, we will call the theory at $R = 1$ the "covering space" theory and that at $R = r$, the orbifold space theory.

### 3.2 Constructing Virasoro orbifold boundary state

In constructing an orbifold boundary state, one first identifies the physical parameter in which the orbifold group action $\Gamma$ acts. Then the "untwisted" or bulk orbifold boundary states can be constructed from the covering space by summing over all images of the boundary state under the orbifold group action and dividing the sum by the normalization factor, $\sqrt{|\Gamma|}$ . This prescription ensures that the resulting boundary state is $\Gamma$ invariant as the non-invariant closed string states are effectively projected out. For the simple example of orbifolding an $S^1$ theory by the $\mathbb{Z}_2$ shift action $X \to X + 2\pi R/2$, the D0-brane bulk orbifold boundary state is given by $\frac{1}{\sqrt{2}}\left(|D0\rangle\rangle_{x_0} + |D0\rangle\rangle_{x_0+2\pi R/2}\right)$. Here, the parameter which is acted upon is the location $x_0$ of the D0-brane.

For orbifolding to the $R = r$ theory, we note that the boundary states at $R = 1$ are parameterized by $(\lambda, \phi_1, \phi_2)$. In changing the radius $R$, the location of the D0-brane, $\phi_1$, and the value of the constant gauge potential, $\phi_2$, must now satisfy new periodicities. The orbifold group element $h_{m',n'}$ in (3.1) acts on the parameters as follows: $\phi_1 \to \phi_1 + 2\pi r m'$ and $\phi_2 \to \phi_2 + 2\pi n'/r$ . $\lambda$ being the coupling strength of the tachyon potential is invariant under the orbifold action. The Virasoro orbifold boundary state can then be expressed as

$$\begin{aligned} |g(\lambda,\phi_1,\phi_2)\rangle\rangle' &= \frac{1}{\sqrt{|\Gamma|}} \sum_{m',n'} \left| g\left(\lambda, \phi_1 + 2\pi r m', \phi_2 + 2\pi \frac{n'}{r}\right)\right\rangle\!\!\Big\rangle \\ &= \frac{1}{2^{\frac{1}{4}}\sqrt{|\Gamma|}} \sum_{m',n'} \sum_{j,m,n} (-1)^{j-n} \times \\ &\qquad \times D^j_{m,n}\left[ g\left(\lambda, \phi_1 + 2\pi r m', \phi_2 + 2\pi \frac{n'}{r}\right)\right] |j;m,n\rangle\rangle \,, \end{aligned} \tag{3.2}$$

where the sums over $m'$ and $n'$ is from zero to $m'_{\rm max}$ and $n'_{\rm max}$, respectively, with $(m'_{\rm max} + 1)(n'_{\rm max} + 1) = |\Gamma|$. Note that $|\Gamma|$ is determined by $r$ and is independent of $\lambda$ . For $r = M/N$, (3.2) is equivalent to the projection prescription of the Virasoro boundary states given in [2]. In a different framework, Friedan [1] has also proposed a projection mechanism in describing the space of $c = 1$ boundary conditions.

We point out that the bulk orbifold boundary states at $R = r$ in (3.2) are expressed only in terms of closed string states already present in the $R = 1$ theory. There are also fundamental orbifold boundary states that utilize twisted sector closed strings. These are called fractional boundary states (see for example [17, 18] and references therein). For the asymmetric orbifold (3.1), they correspond to the the well-known single D0- or single D1-brane boundary state in the orbifold space and will not be much discussed below.[5] In the next subsection, we will make some general remarks concerning the orbifold construction independent of the radius $r$. We will then proceed to provide some geometrical intuition for the orbifold boundary states on the orbifold space.

### 3.3 Consistency of the orbifold boundary states and boundary entropy

The content of (3.2) is that it expresses Virasoro boundary state of the $R = r$ theory in terms of a linear combination of Virasoro boundary states of the $R = 1$ theory. Therefore, one should check that the resulting boundary state consists only of the Virasoro Ishibashi

[5]The expression for a single D0-brane boundary state at arbitrary radius can be found in subsection 3.4. That for a single D1-brane boundary state can be obtained by T-dualizing the single D0-brane expression.



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states present at the $R = r$ theory. This is easily demonstrated by using the relations

$$D^j_{m,n}(\lambda, \phi_1 + \varphi, \phi_2) = e^{i\varphi(m+n)} D^j_{m,n}(\lambda, \phi_1, \phi_2)\,, \tag{3.3}$$

$$D^j_{m,n}(\lambda, \phi_1, \phi_2 + \varphi) = e^{-i\varphi(m-n)} D^j_{m,n}(\lambda, \phi_1, \phi_2)\,. \tag{3.4}$$

The sum over $m'$ and $n'$ in (3.2) together with (3.3) and (3.4) imply that the Virasoro Ishibashi state $|j; m, n\rangle\rangle$ will have a non-zero contribution if and only if

$$(m + n)r = Z_1 \qquad \text{and} \qquad \frac{(m - n)}{r} = Z_2\,, \tag{3.5}$$

where $Z_1, Z_2 \in \mathbb{Z}$. Now recall that $2m$ and $2n$ are respectively the momentum zero modes $p_L$ and $p_R$ in the discrete state primary $\phi^j_{m,n}$. Substituting $(m, n) = (p_L/2, p_R/2)$ in (3.5), we arrive at the condition that $(p_L, p_R) = (Z_1/r + Z_2 r, Z_1/r - Z_2 r)$, precisely the expression for the momentum modes at radius $r$. Therefore, the summation in (3.2) intrinsically projects out closed string states not in the $R = r$ theory.

The orbifold boundary states must also satisfy Cardy's condition [10]. This is the requirement that the tree level closed strings exchange between any two boundary states, under modular transformation, can be expressed as the annulus partition sum of open strings in terms of integer combination of conformal characters. Given that the $R = 1$ boundary states satisfy Cardy's condition, the orbifold boundary states as a set also satisfy Cardy's condition. Without the normalization factor on the r.h.s. of (3.2), Cardy's condition is trivially satisfied since an orbifold boundary state is then just a summation of $R = 1$ boundary states. The normalization factor just simply reduces the redundancy in the open string conformal characters due to the summation. More explicitly, the closed string tree amplitude between two boundary states $|g\rangle\rangle$ , $|g'\rangle\rangle$ at $R = 1$ is expressed in terms of the open string partition sum as [2]

$$\langle\langle g\left(\lambda', \phi_1', \phi_2'\right) | q^{\frac{1}{2}(L_0 + \tilde{L}_0 - \frac{c}{12})} | g\left(\lambda, \phi_1, \phi_1\right)\rangle\rangle = \sum_{n\in\mathbb{Z}} \frac{\tilde{q}^{\,(n+\frac{\alpha}{2\pi})^2}}{\eta(\tilde{q})}\,, \tag{3.6}$$

where $\eta(\tilde{q})$ is the Dedekind $\eta$-function and $\alpha$ is given by

$$\cos\alpha = \cos\lambda \cos\lambda' \cos(\phi_1 - \phi_1') + \sin\lambda \sin\lambda' \cos(\phi_2 - \phi_2')\,. \tag{3.7}$$

We see that the open string conformal character for a given $\lambda$ and $\lambda'$ is only dependent on the differences, $\phi_1 - \phi_1'$ and $\phi_2 - \phi_2'$. This leads to an overall multiplicity in the conformal characters for orbifold boundary states which is divided out by the inclusion of the normalization factor. One still has to check that Cardy's condition is satisfied when the fundamental fractional boundary states containing twisted sector closed string states are also considered. Indeed, this is the case as can be explicitly checked by noting that the closed string twisted sector states do not couple to the Virasoro boundary states and by using equations (3.2), (3.3), and (3.4).

It is also worthy to note that the boundary entropy $g_b$ [19], is identical for all Virasoro boundary states at a given radius. This is as expected since these boundary states are connected by turning on truly marginal boundary fields, and therefore, the boundary entropy remains constant [19, 20]. The boundary entropy corresponds to the coefficient of the vacuum state $|p_L = 0, p_R = 0\rangle$. Since the matrix element $D^0_{0,0}(g) = 1$ for any $g$, we have that

$$g_b = \frac{1}{2^{\frac{1}{4}}\sqrt{|\Gamma|}} \sum_{m',n'} 1 = \frac{\sqrt{|\Gamma|}}{2^{1/4}} \,. \tag{3.8}$$

Note that for rational $r = M/N$, $g_b = \sqrt{MN}/2^{1/4}$. In contrast, for irrational $r$, the boundary entropy for all Virasoro boundary states is infinite. This results from the orbifold action $\Gamma$ having infinite order at irrational $r$. We will give a geometrical interpretation of this infinity in the next subsection.

### 3.4 Geometry of orbifold boundary states

In this section, we develop some intuition on the geometry of the asymmetric orbifold boundary states (3.2) in the orbifold space. Although it is believed that the Virasoro branes (i.e. $|g\rangle\rangle$ with $0 < \lambda < \pi/2$) are fundamental, we do not have a good geometrical picture describing a D1-brane with a marginal tachyon potential turned on. For clarity of description, we will mostly consider the geometry of the orbifold D0-brane boundary state ($\lambda = 0$) for arbitrary radius $r$. The description of the orbifold D1-brane state ($\lambda = \pi/2$) is identical to that of the D0-brane after a T-duality transformation. In focusing on D0-branes, the Virasoro Ishibashi states will not play a role in the description. Therefore, our intuition can be applied to our orbifold with the covering space taken to have radius $R \neq 1$. We will develop our understanding by analyzing sequentially four generic cases: $(a)$ $r = 1/N$ ; $(b)$ $r = N$ ; $(c)$ $r = M/N$ ; $(d)$ $r =$ irrational.

$(a)$ $r = 1/N$ case

This is the simplest case with the orbifold being symmetric. The orbifold D0-brane boundary state is a single D0-brane. (3.2) expresses it as a sum of its $N$ images in the $R = 1$ covering space. A D0-brane has boundary entropy $g_b = 1/(2^{1/4}\sqrt{R})$. Thus, for $R = 1/N$, $g_b = \sqrt{N}/2^{1/4}$ which is exactly that given in (3.8) for $|\Gamma| = N$. The orbifold of the D1-brane, $\lambda = \pi/2$, is more interesting and is described by that of the D0-brane for $r = N$.

$(b)$ $r = N$ case

Here, the parameter space of $x_0$ has increased in size by $N$-fold such that $x_0 \sim x_0 + 2\pi N$. The asymmetric orbifold effectively creates copies of the original parameter space instead of identifying the covering space. In doing so, the orbifold D0-brane boundary state should describe a configuration of $N$ evenly-spaced D0-branes in the $R = N$ theory. Indeed, this is the case as can be explicitly checked using (3.2) and the general boundary state expression for a single D0-brane,

$$\begin{aligned} |D0\rangle\rangle_{x_0} &= \frac{1}{2^{\frac{1}{4}}\sqrt{R}} \sum_{n\in\mathbb{Z}} e^{i\,\frac{n}{R}x_0} e^{\sum_{m=1}^{\infty}\frac{1}{m}\alpha_{-m}\tilde{\alpha}_{-m}} \left| p_L = \frac{n}{R}, p_R = \frac{n}{R} \right\rangle \\ &\equiv \frac{1}{2^{\frac{1}{4}}\sqrt{R}} \sum_{n\in\mathbb{Z}} e^{i\,\frac{n}{R}x_0} \left| \frac{n}{R}, \frac{n}{R} \right\rangle\!\!\Big\rangle_D \,, \end{aligned} \tag{3.9}$$

where in the second line we have written it in terms of U(1) Ishibashi states. Again, the boundary entropy $g_b = N\frac{1}{2^{1/4}\sqrt{N}}$ is precisely that for $N$ D0-branes at $R = N$.

As an aside, we use this example to raise a subtle issue concerning the orbifold boundary state construction. The expression for the D0-brane orbifold boundary state as given in the l.h.s. of (3.2) is an explicit sum of $N$ evenly-spaced D0-brane boundary states on the orbifold space at $R = N$. However, to denote a superposition of fundamental boundary states, one should technically utilize the Chan-Paton (CP) index instead of explicitly summing over boundary states. A boundary state contains information on the couplings of closed string fields to the open string identity fields [21, 8, 11].[6] Only those closed string states that appear in the boundary state have a non-zero coupling. Ishibashi states that are present in the fundamental boundary states may be canceled out in the summation process as for instance in the $N$ D0-branes configuration. This misleadingly suggests that certain closed string fields do not couple to open string identity fields and thus leads to the violation of the cluster condition, one of the sewing constraints [22] required for a local boundary conformal field theory (BCFT). With the use of the CP index, all couplings of closed string fields to the open string identity fields are explicitly present in the boundary state and the cluster condition can be verified. This subtlety can be overlooked if the boundary state is used solely to calculate the open string spectrum and test Cardy's condition. Thus, we see that the orbifold mechanism functions at the level of summation and does not distinguish whether the orbifold boundary state is a fundamental or a superposition of fundamental states.

$(c)$ $r = M/N$ case

The geometry of the orbifold D0-brane boundary state in this case is made trivial by following the descriptions of the two preceding cases. One can first orbifold by $r = 1/N$ followed by another orbifold by $r = M$. The net result on the $R = M/N$ orbifold space is $M$ equally-spaced D0-branes with a separation of $\Delta x_0 = 2\pi/N$. Notice that on the orbifold space, the original shift symmetry in the $R = 1$ theory, $x_0 \to x_0 + 2\pi$, is still present - that is there is a D0-brane located a distance $\pm 2\pi$ apart from any single D0-brane. Indeed, the original symmetry of the theory is preserved under the orbifold.[7] This implies the presence of conformal dimension one winding states that stretch a distance of $2\pi$. In addition to the translation field $i\partial X$, the marginal winding states together provide two additional truly marginal boundary fields that can be used to deform the boundary state.[8] A deformation by the truly marginal winding fields will lead to the orbifold Virasoro boundary state

[6] A field with zero conformal dimension is by definition an identity field. In general, for a superposition of fundamental boundary states, the associated open string theory has more than one identity fields.

[7] This becomes manifest by working in the $\mathbb{R}^1$ covering space of $S^1$. A D0-brane situated at $x_0 = 0$ in the $R = 1$ theory corresponds to a D0-brane located at each point $2\pi\mathbb{Z}$ on $\mathbb{R}^1$. Applying the orbifold action, D0-branes in $\mathbb{R}^1$ are now located at $2\pi(m+rn)$ for $m, n \in \mathbb{Z}$. This configuration corresponds in the orbifold space to D0-branes located at $2\pi(m + rn) \bmod 2\pi r = 2\pi m \bmod 2\pi r$, which preserves the original $2\pi$ shift symmetry.

[8] It is worth noting that a conformal dimension one winding field that stretches a distance $2\pi$ by itself is



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which from (3.6) and (3.7) can be seen to have three open strings fields of dimension one. The presence of truly marginal fields allows all bulk orbifold boundary states to be connected by marginal deformations.

(*d*) $r = irrational$ case

The group action $\Gamma$ for irrational $r$ is of infinite order and the orbifold boundary states given in (3.2) is exactly those given in [2, 3]. It is interesting to point out that the orbifold boundary state only consists of closed string states from the conformal family of the identity field which are present in the closed string theory at any radius. As for the geometry of the orbifold D0-brane boundary state, it must preserve the symmetry of the original shift symmetry. This implies that the orbifold boundary state at $R = r$ is geometrically that of D0-branes located at points $p = 2\pi m \bmod 2\pi r$. This gives a dense set of D0-branes and explains why the boundary entropy, that is after "summing" over the contributions from the infinite number of D0-branes present, is infinite. More significantly, the Virasoro boundary states, obtained by deforming with truly marginal boundary fields and suspected to be fundamental, must also have infinite boundary entropy. Since the mass of the brane is proportional to the boundary entropy [23], we find that the irrational radius Virasoro boundary states have infinite mass, and therefore, are not likely to be physically relevant.[9]

## 4. Boundary State Orbifold as Open String Quotient

We have applied an asymmetric orbifold on the boundary states at $R = 1$ to obtain boundary states at $R = r$. This is a closed string description mapping boundary conditions from an open string theory at a particular radius to those at a different radius by means of an orbifold. As might be expected, there is an open string description too. In the open string sector, different boundary conditions are distinguished by their associated vertex operator algebras. The boundary orbifold mechanism suggests that it is possible to generate the vertex operator algebras at $R = r$ from those at $R = 1$. Indeed, this can be accomplished using the quotient algebra construction [25, 26] which has been used successfully to describe open strings on symmetric orbifolds [27, 28].

To describe the vertex operator algebra at $R = r$, we need to study how the group action $\Gamma$ acts on the vertex operators $V(X, \tilde{X})$ at $R = 1$. Of course, $\Gamma$ must act on the fields, $X$ and $\tilde{X}$, as given in (3.1). But notice that in (3.2), $\Gamma$ maps out the images of the orbifold boundary state in the covering space. With a superposition of boundary states in the covering space, the open string sector covering space algebra $V(X, \tilde{X})$ must be matrix-valued and labelled by two CP indices. Thus, $\Gamma$ will act on the CP index with an action

not truly marginal (see [8] for conditions on true marginality). In fact, it is precisely because the D0-branes are equally-spaced that we can construct two truly marginal winding fields. The two are linear combinations of all dimension one winding fields.

[9]One may wonder what happens to the finite mass Virasoro boundary states at rational radius when perturbed by the marginal closed string vertex operator $\partial X \bar{\partial} X$ to irrational radius. Indeed, the Virasoro boundary condition under this perturbation becomes non-conformal. For $R = 1/2$, Sen has explicitly shown in [24] that a non-zero tadpole arises at first order in perturbation.



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$\gamma(h)$ dependent on the representation space chosen for the CP indices. For obtaining the bulk boundary states, such as the Virasoro orbifold boundary states of (3.2), the regular representation is required with CP indices $i, j = 0, \ldots, |\Gamma| - 1$. A valid representation with a smaller dimension will give the vertex algebra for a fractional boundary state. The quotient algebra or the algebra of vertex operators at $R = r$ is then the matrix-valued algebra $V(X, \tilde{X})$ satisfying the equivalence relation condition

$$\gamma(h)\, V(X, \tilde{X})\, \gamma^{-1}(h) = V(h(X, \tilde{X})) \qquad \text{for all } h \in \Gamma\,. \tag{4.1}$$

The quotient algebra is in fact universal in that it may be applied to any group action $\Gamma$ with an unitary representation $\gamma(h)$ acting on a Hilbert space $\mathcal{H}$.

### 4.1 Example: $r = 2$

We will work out the vertex algebras for D1- and D0-brane boundary conditions for $r = 2$ and will not consider non-U(1) boundary conditions. The quotient algebra reproduces the $R = 2$ vertex algebra of a single D1-brane for the D1-brane boundary condition and two equally-spaced D0-branes for the D0-brane boundary condition as required from the results of the boundary state orbifold (3.2). For simplicity, we will focus the analysis on the open string tachyon vertex operators $T(X, \tilde{X})$. Since the oscillator modes are not affected by the orbifold, our analysis can be easily extended to include all vertex operators. Before proceeding, we note that open string vertex operators are situated at the boundary $z = \bar{z}$, and in particular, those of the tachyons have the form $e^{i\frac{n}{R}X(z,\bar{z})}$ and $e^{imR\tilde{X}(z,\bar{z})}$, respectively, for a D1$-$ and a D0-brane at radius $R$. We emphasize that $X$ and $\tilde{X}$ have different mode expansions for different boundary conditions. The operator product algebra can be obtained by applying Wick contractions with brane specific Green's functions: $< X(z_1,\bar{z}_1)X(z_2,\bar{z}_2) >_{D1} = < \tilde{X}(z_1,\bar{z}_1)\tilde{X}(z_2,\bar{z}_2) >_{D0} = -\frac{\alpha'}{2}\ln|z_1 - z_2|^2 - \frac{\alpha'}{2}\ln|z_1 - \bar{z}_2|^2$.

We first analyze the D1-brane case. For the quotient algebra, we have $|\Gamma| = 2$ and the nontrivial action on the CP index is given by $\gamma(h_{0,1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \gamma$. Here, the regular representation is the minimal representation because the action $h_{0,1}$ (defined in (3.1) ) on the covering space maps a D1-brane to another D1-brane with a distinct value of the constant gauge potential. The tachyon vertex operator at $R = 1$ is a $2{\times}2$ matrix with vertex operators of the form $e^{inX}$ for diagonal elements and $e^{i(n+\frac{1}{2})X}$ for off-diagonal elements. The half-integer momenta are required for the off-diagonal elements since these are open strings stretching between two D1-branes "separated" by $\Delta\tilde{x}_0 = \pi$. Notice also that $\Gamma$ does not act on the field $X$ in the vertex operator. All together, the tachyon vertex operators that satisfies (4.1) are expressed as

$$T(X) = \sum_n a_n e^{inX} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_n b_n e^{i(n+\frac{1}{2})X} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.2}$$

where all $a$'s and $b$'s denote constant coefficients. Indeed, (4.2) is algebraically equivalent to the expression for the tachyon vertex operator of a single D1-brane at $R = 2$, $\sum_n a'_n\, e^{i\frac{n}{2}X}$.

We now consider the D0-brane case. The nontrivial group action $h_{0,1}$ only acts on the vertex operator by shifting $\tilde{X}$ and does not change the location of the D0-brane. Keeping the regular representation with $\gamma(h_{0,1}) = \gamma$, we have on the covering space two coincident D0-branes. Thus, the tachyon vertex operator at $R = 1$ have the form $e^{im\tilde{X}}$ for both diagonal and off-diagonal elements. The equivalence relation (4.1) requires the general expression for the operator to be

$$\begin{aligned} T(\tilde{X}) &= \sum_m a_m \begin{pmatrix} e^{im\tilde{X}} & 0 \\ 0 & e^{im(\tilde{X}+\pi)} \end{pmatrix} + \sum_m b_m \begin{pmatrix} 0 & e^{im\tilde{X}} \\ e^{im(\tilde{X}+\pi)} & 0 \end{pmatrix} \\ &= \sum_m \sum_{k,l=0,1} a_{m\,k\,l} e^{i(2m+k)\tilde{X}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^l , \end{aligned} \tag{4.3}$$

where a matrix raised to the zeroth power is taken to be the identity matrix. In the second line we have expressed the tachyon operator in the natural matrix basis. This quotient algebra should be compared to that of two equally spaced D0-branes at $R = 2$ which can be expressed as

$$\begin{aligned} T'(\tilde{X}) &= \begin{pmatrix} \sum_m a'_m e^{i2m\tilde{X}} & \sum_m b'_m e^{i(2m+1)\tilde{X}} \\ \sum_m c'_m e^{i(2m+1)\tilde{X}} & \sum_m d'_m e^{i2m\tilde{X}} \end{pmatrix} \\ &= \sum_m \sum_{k,l=0,1} a'_{m\,k\,l} e^{i(2m+l)\tilde{X}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^l , \end{aligned} \tag{4.4}$$

where in the second line, the tachyon operator is expressed in the natural basis of the quotient algebra. Comparing (4.3) with (4.4), we see that both have the same open string spectrum and indeed are equivalent representations of the same algebra. An explicit relationship that connects the two representations is given by the identification $a_{m00} = a'_{m00}$, $a_{m01} = a'_{m10}$, $a_{m10} = a'_{m01}$, and $a_{m11} = -a'_{m11}$. Thus, the quotient algebra and the vertex algebra on the orbifold space are identical up to a change of basis.

For the D0-brane boundary condition, it is also possible to express $\gamma(h_{0,1})$ in the one-dimensional irreducible representation. The irreducible representation can be used here because the location of the D0-brane is invariant under $\Gamma$ in the $R = 1$ covering space. The equivalence condition (4.1) for a single irreducible representation becomes $T(\tilde{X}) = T(\tilde{X} + \pi)$. The quotient algebra thus have elements $T(\tilde{X}) = \sum_m a_m\, e^{i2m\tilde{X}}$, which are exactly the elements of the tachyon vertex algebra for a single D0-brane in the $R = 2$ theory. Thus, we see that for $r = 2$, the irreducible representation corresponds to the fractional brane in the orbifold theory.

### 4.2 Generic radius and the noncommutative Weyl algebra

For a generic $r$, the quotient algebra constructed using the regular representation with dimension $|\Gamma| = (m'_{\max} + 1)(n'_{\max} + 1)$ will produce the expected open string vertex operator algebra for the orbifold boundary state in (3.2). For the D1- and D0-brane boundary conditions, the generalization of the above $r = 2$ example is straightforward. Below, we will describe some general features of the quotient algebra applicable for these two boundary conditions.

For D1- and D0-brane boundary conditions, the vertex operators have dependence only on $X$ and $\tilde{X}$, respectively. Moreover, with the group action being a tensor product, $\Gamma = \mathbb{Z}_{(m'_{\max}+1)} \times \mathbb{Z}_{(n'_{\max}+1)}$ , it is natural to express the regular representation labelled by $j$ as a direct product representation labelled by two indices $j = (j_1, j_2)$ with $j_1 = 0, \ldots, m'_{\max}$ and $j_2 = 0, \ldots, n'_{\max}$. The regular representation corresponds to $|\Gamma|$ number of branes on the covering space. These branes are labelled by $(j_1, j_2)$ and the role they play in constructing the quotient algebra can be gleaned from the actions of $\mathbb{Z}_{(m'_{\max}+1)}$ and $\mathbb{Z}_{(n'_{\max}+1)}$ and from insights obtained from the $r = 2$ example.

Let us consider the D0-brane boundary condition. The $\mathbb{Z}_{(m'_{\max}+1)}$ action (acting on $X$) generates $m'_{\max} + 1$ equally-spaced D0-branes on the covering space. Thus, $j_1$ labels D0-branes that are separated on the covering space. Similar to the D1-brane boundary condition for $r = 2$, the separated branes provide for the open string spectrum of a single D0-brane on the orbifold space. In contrast, the $\mathbb{Z}_{(n'_{\max}+1)}$ action (acting on $\tilde{X}$) does not affect the location of the D0 branes; therefore, $j_2$ labels the $n'_{\max} + 1$ coincident D0-branes situated at each image point. Like the D0-brane boundary condition for $r = 2$, $n'_{\max} + 1$ coincident D0-branes on each image point on the covering space will give, after applying the equivalence relation, the vertex algebra of $n'_{\max} + 1$ equally-spaced D0-branes on the orbifold space. Having only one D0-brane on each image point on the covering space corresponds to only one D0-brane on the orbifold space. Thus, we note that the fractional brane for the D0-brane boundary condition is associated with the representation with dimension $m'_{\max} + 1$. The analysis for the D1-brane boundary condition is identical to that of the D0-brane except for exchanging $m'_{\max} \leftrightarrow n'_{\max}$ and $j_1 \leftrightarrow j_2$.

The quotient algebra for D1- and D0-brane boundary conditions in the direct product representation $(j_1, j_2)$ is manifestly a direct product algebra. For ease of discussion, we will focus on the tachyon vertex operator algebra and treat it as a matrix algebra by ignoring the contribution from the Green's function. Again, we first consider the D0-brane boundary condition. The subalgebra associated with the $j_1$ index is commutative and similar to that given in (4.2). It is generated by the shift matrix $V|j_1 >= |j_1 - 1 >$ with $V$ raised to the $(m'_{\max} + 1)$-th power being the identity matrix. The subalgebra associated with the $j_2$ index like that of (4.3) is noncommutative and is generated by both the shift matrix $V$ and the clock matrix, $U|j_2 >= w^{j_2}|j_2 >$ with $w = e^{i\frac{2\pi}{r}}$. Indeed, the subalgebra labelled by $j_2$ is the Weyl algebra, $UV = w^{-1}VU$.[10] As an explicit example, the tachyon vertex operator satisfying the equivalence relation (4.1) for $r = M/N$ can be expressed as

$$T(\tilde{X}) = \sum_{m} \sum_{n=0}^{m'_{\max}} \sum_{k,l=0}^{n'_{\max}} a_{m\,n\,k\,l}\, e^{i\,nr\tilde{X}} V'^{n} \otimes\; e^{i(Mm+k)\tilde{X}} U^{k}\; V^{l}\,, \tag{4.5}$$

where $V'$ denotes the shift operator acting on the $j_1$ index and $U$ and $V$ act on the $j_2$ index. With $j_2 = 0, \ldots, n'_{\max}$, the Weyl subalgebra is present only if $n'_{\max} \geq 1$. Hence, on the orbifold space, there must be two or more equally-spaced D0-branes. For the D1-brane

[10] The presence of a noncommutative algebra at irrational radius $R$ was hinted at by Friedan [1]. Also, the appearance of the Weyl algebra in quotient algebras has been much exploited in the field of noncommutative geometry [29]. See, for example, [25] for a concise exposition from a physical perspective.

boundary condition, the Weyl subalgebra is associated with the $j_1$ index with $w = e^{i2\pi r}$. In general, the Weyl subalgebra is present only if the quotient algebra is associated with an orbifold boundary state that is not fundamental (i.e. not a Virasoro brane nor a single D0- or D1-brane).

The appearance of the Weyl algebra can be understood from the orbifold space perspective. As we have argued, the quotient algebra is identical to the vertex algebra associated with the orbifold boundary state. This implies that the vertex algebra for two or more equally-spaced D0- or D1-branes contains the Weyl algebra. Indeed, we have seen this in the $r = 2$ example, where the vertex operators of two equally-spaced D0-branes can be expressed in the clock and shift basis as in (4.4). With equally-spaced branes, each element in the matrix vertex algebra and those diagonally above and below it have the same set of allowed momentum. Because of this, one can always express the vertex operator matrix using the complete matrix basis generated by $U$ and $V$. In this basis, the Weyl algebra becomes manifest.

## 5. Discussion

The asymmetric shift orbifold that we have presented in fact connects two arbitrary points on the moduli space of the closed string theory on a circle. We have chosen the covering theory to be situated at $R = 1$ only to facilitate the construction of the Virasoro boundary states at other radii. In doing so, all Virasoro boundary states from the orbifold are bulk boundary states and can be easily constructed without considering twisted sector closed string states. As noted, in the open string description, bulk boundary states correspond to the regular representation in the quotient algebra construction.

Bulk orbifold boundary states are interesting objects. Neglecting the normalization factor $\sqrt{|\Gamma|}$, the boundary state is present in both the covering and the orbifold theory. The bulk orbifold boundary state in the *covering theory* is a superposition of boundary states; in the orbifold theory, we have seen that it may be either a fundamental boundary state or also a superposition of boundary states. That it can be found within two different closed string theories is the direct result that the spin zero closed string states that make up the boundary state are present in both closed string theories. Indeed, the orbifold action has provided us a systematic mechanism to identify the set of shared closed string states between the two theories. Since each boundary state corresponds to a BCFT, a bulk orbifold boundary state effectively relates two BCFTs situated within two different bulk theories.

We speculate that a generalization of the bulk orbifold boundary state may be made even when two closed string theories are not connected by an orbifold but share a set of spin zero closed strings. We point out that boundary states are typically constructed using only a subset of the set of spin zero closed string states. Thus, in general, consider two closed string theories, labelled by $A$ and $A'$, that have in common a set of states called $S$. Now if a boundary state $|B\rangle\rangle$ that satisfy Cardy's condition (at least with itself) can be constructed from states in $S$ in theory $A$, then $|B\rangle\rangle$ up to a normalization is also a boundary state in theory $A'$. The presence of $|B\rangle\rangle$ effectively gives a relationship between two boundary



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field theories associated with different bulk field theories. Specifically, the boundary field spectrum of the two BCFTs, possibly up to an integer multiplicity factor, are identical by Cardy's condition. This generalization raises the possibility that two BCFTs may be related although there may not be a relation between their associated bulk conformal field theories.

We should mention that throughout the paper, we have neglected the sewing constraints that must be satisfied for the local consistency of any BCFT. We will only add that one of the sewing constraints, the cluster condition, may be helpful in determining whether an orbifold boundary state is fundamental. As noted earlier, the construction of orbifold boundary states is at the level of "summation" and does not provide any information on whether the orbifold boundary state is fundamental or a superposition of fundamental boundary states. However, if the boundary state does not satisfy the cluster relation, then it can not be fundamental.

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